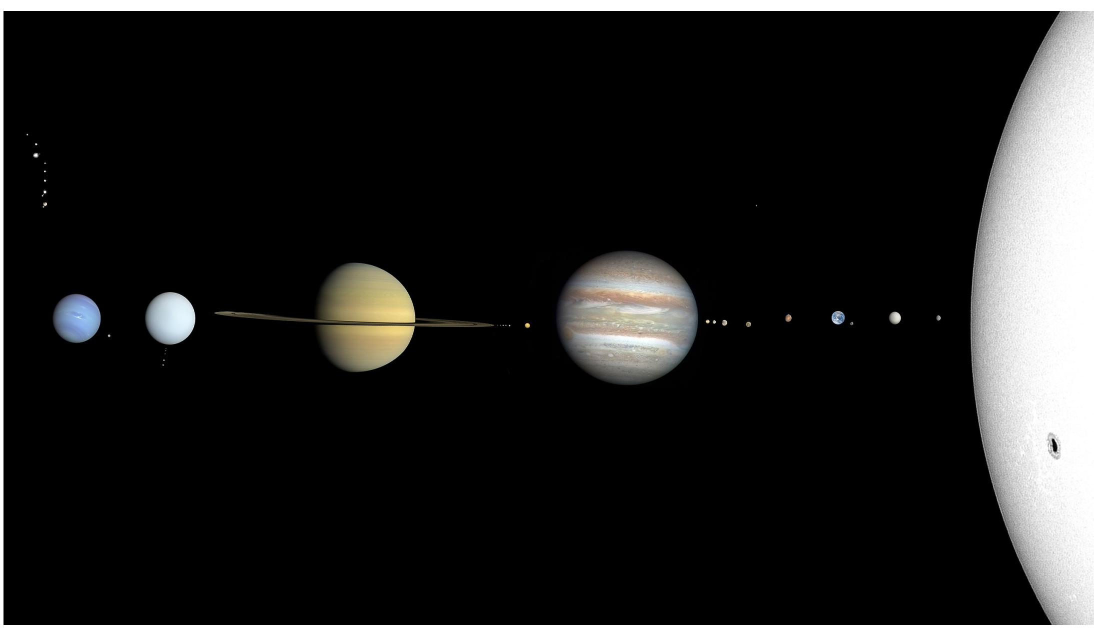
Grade 11-12 Math Circles
Nov. 9, 2022
The Mathematics of Climate Change 2

Introductory Comments

Last week we discussed some basics of using mathematics to describe the temperature, and in the slides and the discussion of them we learned a bit about Climate Change itself. One central feature of climate change is its exponential nature; hard to see for some time, then grows explosively (at least as far as our perception is concerned). This week we are going to talk about climate in its not changing very much state, or in other words before man made climate change and on scales shorter than those associated with ice ages and the like. Here the goal is to do some mathematics that makes contact with science. In particular we want to talk about **energy**. Energy is an abstract quantity in the sense that we cannot see energy. We can experience it through its manifestations, for example putting your hand above a candle lets you feel how thermal energy in combustion leads to an increase of temperature above the flame. Energy for the climate system comes from the Sun, which is a giant astronomical object (see the figure below) that dwarfs the tiny Earth. The output from the Sun does vary, but not in a significant way (at least for the level of description in this lesson), and this radiation drives the climate system on Earth. The task for this lesson is to learn some basics of radiative transfer and use it to make very, very simple models of climate.



This image shows the Solar System and the Sun. You can see the relative sizes of the planets, with the Earth tiny compared to the Gas Giants. Of course, the Sun dwarfs all others and is the only energy emitting body. While the Sun is far from the Earth, its immense size and efficiency of the fusion within it, gets enough energy to Earth to make for a livable climate. A basic model of this is our goal in this and the next lecture.

Stop and Think

The basic law relating temperature and radiation is called Stefan's law for a so–called black body (which radiates all possible energy). It reads

$$S = \sigma T^4$$

where S is the power radiated per square meter, T is the temperature in degrees Kelvin (so that water freezes at 273 K), and σ is a universal constant

$$\sigma \approx 5.6704 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$$

This means that the power radiated depends quite strongly on temperature. The Sun is said to have a typical temperature of about $5,722$ K, while the Earth has a typical temperature of about 300 K. This means the Sun radiates an astonishing 1.32×10^5 as much energy as the Earth. We are used to linear laws, and it is worth thinking about how different a law involving a fourth power behaves (see Exercise 1 below).

Thinking about Stefan's Law and its application

Stefan's law is interesting in itself, but it becomes far more interesting if it is used as part of the exercise of mathematical modelling. Mathematical modelling is the exercise of creating and solving a "cartoon" of the real world. Ideally this cartoon captures the essence of what we want to model and is easy to use. In the presentation for this week I will describe some aspects of climate modelling, which involves some pretty fancy (and very complex) models. Since we want to do our modelling by hand we will stick to really simple models; if you want to pursue the "cartoon" metaphor we are talking stick man as opposed to Pixar.

Our modelling starts with the notion of a black body. A black body absorbs electromagnetic a portion of coming in and reflects the rest. The fraction of energy reflected is called the **albedo**.

There are many detailed scientific studies of albedo, but the one you may recall from grade school is that dark objects absorb much more heat than white objects. This means that ice reflects, while ice free regions absorb heat. Albedo is a parameter between zero and one, and we will denote it by the Greek letter α. Standard tables give road asphalt as about $\alpha = 0.05$ and that of fresh snow as about $\alpha = 0.8$. The energy that is not reflected is assumed to become evenly distributed in the black body (this is manifestly untrue for the real planet; you aren't going to go on a beach vacation to Greenland!) and whatever temperature is reached, the black body radiates according to Stefan's law at this temperature.

We next assume all energy for our black body Earth comes from the Sun, so that we an write a so-called **energy balance**, first in words

$$\text{Energy from Sun absorbed} - \text{ Energy radiated out} = 0$$

and now as an equation

$$S(1-\alpha) - \sigma T^4 = 0.$$

Physicists, Meteorologists and many other scientists would refer to this equation as a **Conservation Law** and the study of these laws dominates the quantitative side of natural science.

Solving the Model

I put a section break here to remind the reader that the process of modelling and the process of solving the model are often completely separate exercises. In the context of climate, it could well be that entirely different teams do the modelling and the solving!

If we measure the energy from the Sun and come up some way to estimate the albedo of Earth (a non-trivial exercise given the many different types of surfaces; ice and snow, ocean, jungle, cities, etc) we can rearrange the above to read

$$T = \left(\frac{S(1-\alpha)}{\sigma}\right)^{1/4}$$

As a concrete sample, the value often given for the solar output is $S = 342.5$ W m^{-2}, and $\alpha = 0.3$ for the albedo. The estimate of temperature in Kelvin is thus

$$T \approx 255\text{K}$$

a frosty -18 Celsius.

Activity 1

Forget for the moment that the numerical answer we got was rather unpleasant. Discuss the meaning of the solution for temperature derived above. In particular, how sensitive is the solution to variations in solar output? For the record estimates of solar variation might be ± 1 W m-2.

Discussion: (you can record what we talk about here):

Activity 2

Now let's acknowledge that our "best" model gave an unrealistic Earth. Try to find combinations of parameters that give a black body Earth temperature that is above zero degrees Celsius.

Discussion: (you can record what we talk about here):

Exercise 1

Consider the radiated power as a function of temperature, T,

$$S(T) = \sigma T^4.$$

We have mentioned that this is a nonlinear law. Since we have more experience with linear functions let's figure out how to approximate the above (for those with calculus experience, we are effectively writing the differentials out, without calling them that). First write $T = T_0 + \Delta T$. Next substitute this into the expression and expand in powers ΔT. You should find something like

$$S(T) = S(T_0 + \Delta T) = \sigma T_0^4 + \text{something } \times \Delta T + \text{ something else } \times \Delta T^2 + ...$$

You can stop your calculation after two terms because you next want to consider to be small, or $\Delta T \ll 1$. What does this mean for the comparison between ΔT and ΔT^2? Use this to find a linear approximation

$$S(T_0 + \Delta T) \approx \sigma T_0^4 + \text{something } \times \Delta T.$$

Evaluate the expression at the temperature of the Sun and Earth and explain the difference using the mathematical terminology used to describe lines.

Exercise 1 Solution

Expanding using the binomial theorem gives

$$E(T_0 + \Delta T) = \sigma T_0^4 + 4\sigma T_0^3 \Delta T + 6\sigma T_0^2 \Delta T^2 + ...$$

so that the linear approximation is

$$E(T_0 + \Delta T) \approx \sigma T_0^4 + 4\sigma T_0^3 \Delta T.$$

The key point here is that the slope of the line is $4\sigma T_0^3$, or in other words it depends on the third power of the temperature we are near. So the line that forms the linear approximation is much, much steeper for the Sun compared to that for the Earth.

Exercise 2

This problem is a bit harder. I'd like to use the linearization of the previous problem to discuss the time dependent problem since the real climate most certainly changes with time. The basic law is now written in words as

$$\text{The Rate of Change of the total energy with time} = \\ \text{Energy from Sun absorbed} - \text{ Energy radiated out}$$

and now as an equation

$$c_p \frac{dT}{dt} = S(1-\alpha) - \sigma T^4.$$

Here c_p is a physical constant called the heat capacity.

If we return to the exponential functions of last week, we can use the calculus fact that

$$\frac{d}{dt}\exp(at) = a\exp(at)$$

and we would like to show that the time dependent behaviour will behave like an exponential.

Step 1: We will let $T = T_0 + \Delta T$. If T_0 is a time independent solution write down the equation it satisfies.

Step 2: The rate of change of a constant like T_0 is zero so that

$$\frac{dT}{dt} = \frac{d\Delta T}{dt}.$$

So the left hand side is OK. Substitute into the right hand side using the results of the previous exercise and use Step 1 to simplify.

Step 3: Use the derivative of the exponential given above to identify a for this problem. What does it tell you about how quickly or slowly the climate system adjusts?

Exercise 2 Solution

We have that

$$S(1-\alpha) - sigma T_0^4 = 0$$

Now writing out the time independent equation using the linear approximation gives

$$c_p \frac{d\Delta T}{dt} = S(1-\alpha) - \sigma T_0^4 - 4\sigma T_0^3 \Delta T.$$

But the first two terms on the right hand side cancel, so we get the much simpler

$$\frac{d\Delta T}{dt} = -\frac{4\sigma T_0^3}{c_p}\Delta T$$

which can be matched up with the derivative of the exponential provided that

$$a = -\frac{4\sigma T_0^3}{c_p}.$$

The climate system thus adjusts more quickly the higher the temperature is.